View metadata, citation and similar papers at core.ac.uk

brought to you by CORE

provided by Texas A&M Repository

Spontaneously generated atomic entanglement in free space: reinforced by incoherent pumping

Ling Zhou,[1] Guo Hui Yang,[1] and Anil K. Patnaik[2,3]

[1] *School of Phys. and Optoelectronic Tech., Dalian University of Technology, Dalian 116023, PR China*

[2] *Department of Physics, Texas A&M University, College Station, Texas 77843 USA*

[3] *Department of Physics, Wright State University, Dayton, OH 45435 USA*

(Dated: September 19, 2018)

arXiv:0908.0831v1 [quant-ph] 6 Aug 2009

We study spontaneously generated entanglement (SGE) between two identical multilevel atoms in free space via vacuum-induced radiative coupling. We show that the SGE in two-atom systems may initially increase with time but eventually vanishes in the time scale determined by the excited state lifetime and radiative coupling strength between the two atoms. We demonstrate that a steady-state SGE can be established by incoherently pumping the excited states of the two-atom system. We have shown that an appropriate rate of incoherent pump can help producing optimal SGE. The multilevel systems offer us more chanel to establish entanglement. The system under consideration could be realized in a tight trap or atoms/ions doped in a solid substrate.

PACS numbers: 03.65.Ud, 42.50.Dv

I. INTRODUCTION

The recent development of quantum technologies strives to resolve the quest for the best entanglement source in quantum optical systems. Though entanglement is observed in a variety of systems, entanglement in atomic systems are favored as more scalable and practical systems, compared to their "photonic only" counterparts due to the development of reliable state-of-the-art technologies to control the atoms one-at-a-time [1] which can be precisely scaled to many atoms [2, 3]. Many exciting developments of entanglement sources are involved in atomic systems such as entanglement via atom-cavity coupling [4], atom-atom entanglement via cavity [5], entanglement in trapped ions/atoms [3, 6], atomic entanglement in an optical lattice by controlled collision [7] and also atomic entanglement via external fields [8, 9]. Recently, it has also been shown that well separated atomic ensembles can also be entangled via coherent coupling between them [10]. Scully has extensively discussed entanglement in two, three and many atoms via a single photon [11] and has shown that such an ensemble can produce directional spontaneous emission [12].

Amongst the different atomic entanglement generation processes, an interesting and useful category is spontaneously generated entanglement (SGE) sources via interaction of atoms with a common bath of cavity field [5], vacuum [3, 6, 13], heat bath [14] or even a spin chain [15]. Usually, the baths have very short correlation time and hence potentially cause disentanglement [17] and decoherence [16] in an entangled system. Agarwal and Patnaik [19] have shown that coherences in two multilevel atoms can be generated from their interaction with a common vacuum bath via the retarded dipole-dipole (dd) coupling when they are placed in the close proximity of each other. Effect of such vacuum induced coherence (VIC) on the collective resonance fluorescence is discussed in [20]. SGE is particularly interesting from the application point of view because: practical quantum devices are *often unavoidably coupled* to the environmental bath and hence SGE can occur naturally.

Most of the works listed above are focused on SGE in two-level atoms. Study of multilevel systems are important because in certain situations, participation of additional internal atomic levels in the process of generating entanglement or causing disentanglement is unavoidable. For example, when two atoms, having a triplet P-state as their excited [19] or ground [20] state, are placed in close proximity, i.e., the interatomic distance is less than the wavelength of the atomic transitions involved, $R < \lambda_0$, even the dipoles involving the orthogonal $\sigma_\pm$ transitions can radiatively couple to generate additional coherences. Recently, Keitel and coworkers [21] have explicitly shown that the two-level approximation fails in such a situation. Furthermore, multilevel systems can open up new channels in bath assisted SGE in a vary natural way and even can give more control parameters [18]. To the best of our knowledge, only a few studies addresses entanglement in three-level atoms interacting with a continuum via the radiative coupling [17, 18].

In this paper, we investigate the steady state SGE between two *radiatively coupled and incoherently pumped atoms* having their energy levels in a V configuration with non-degenerate excited states; see Fig. 1. We derive a master equation and trace over the field part to obtain the equations for the atomic dynamics. We obtain an analytical solution to show that multilevel systems are preferable compared to two-level systems for SGE because they add additional coupling channels to enhance the entanglement. We demonstrate that the SGE can be sustained to achieve a steady state entanglement by incoherently pumping the atoms. It may be noted that we are working outside the regime of VIC. In our two-atom system VIC could be generated if the excited states of both the atoms are degenerate or near degenerate [19, 20].

The organization of the paper is the following: In Sec. II we derive a master equation for the two atoms interacting with common vacuum. Tracing over the vacuum bath parameters, we obtain the equations for the system dynamics. In Sec. III, we present the time evolution of the entanglement between the two atoms that occurs only for a short period of time. In Sec. IV, we derive the atomic density matrix equations when atoms are incoherently pumped. We show that a steady state entanglement can be obtained between the two atoms purely via incoherent processes. We summarize and discuss our result in Sec. V.

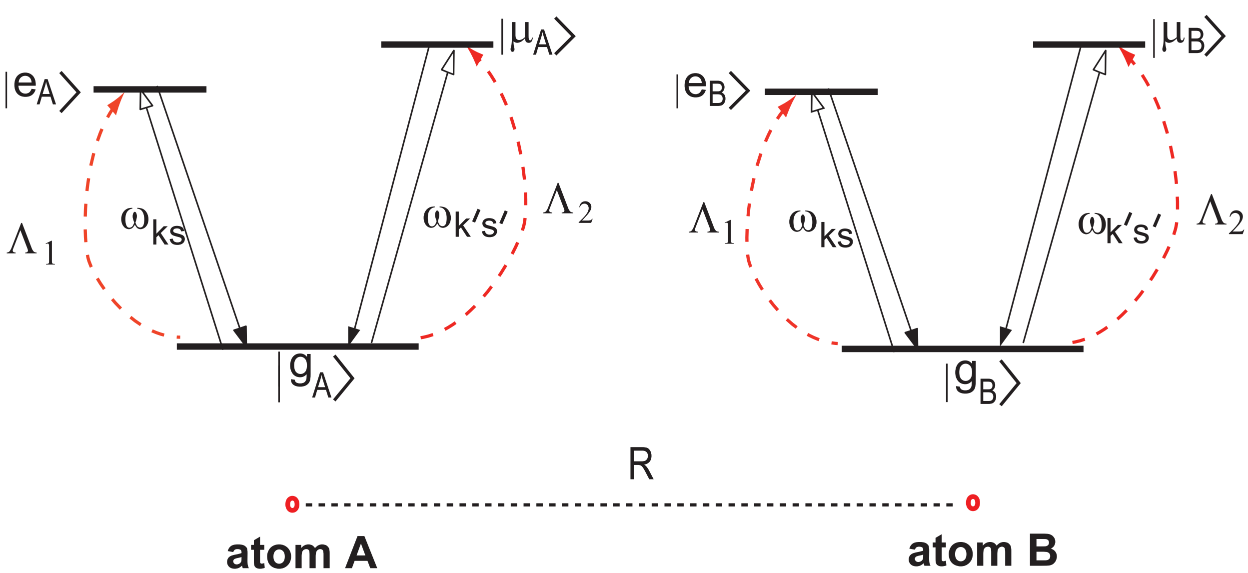

FIG. 1: (Color online) The two identical atoms under consideration. The V type atoms with non-degenerate excited states. The distance between the two atoms is considered to be small compared to the radiation wavelength, $R < \lambda_0$.

II. THE TWO-ATOM SYSTEM AND THEIR DYNAMICS

We consider two identical three level V systems (say A and B) in free space having two excited states $|e_\alpha\rangle$ and $|\mu_\alpha\rangle$ ($\alpha = A, B$), and a ground state $|g_\alpha\rangle$, as depicted in Fig. 1. Both the atoms couple to the same vacuum field. While we do not wish to loose the generality of our results, our scheme can correspond to two ^{40}Ca atoms in a magneto-optical trap (MOT) in presence of a static magnetic field. The ground state can correspond to 4^1S_0 state and the excited states can correspond to the magnetic sublevels 4^1P_1 of Ca atom. The static magnetic field would remove the degeneracy of the 4^1P_1 sublevels and the states $|e_\alpha\rangle$ and $|\mu_\alpha\rangle$ can correspond to $m_l = \pm 1$ levels. Note that we restrict to the situation where the cross couplings between $|e_\alpha\rangle \leftrightarrow |g_\alpha\rangle$ and $|\mu_\alpha\rangle \leftrightarrow |e_\alpha\rangle$ transitions are eleminated by considering the non-degenerate excited states. Thus the photon emitted from $|e_\alpha\rangle \to |g_\alpha\rangle$ ($|\mu_\alpha\rangle \to |g_\alpha\rangle$) can only be absorbed by $|g_\alpha\rangle \to |e_\alpha\rangle$ ($|g_\alpha\rangle \to |\mu_\alpha\rangle$). For simplicity, we consider only the case of real dipole moments for our discussion below. These results can be easily generalized to complex dipoles, e.g., involving magnetic sublevels.

In this section, we derive the system dynamics with only the contributions from the two-atom coupling with vacuum, i.e., in absence of incoherent pumping. The role of incoherent pumping will be discussed in detail in the Sec. IV. The Hamiltonian of the two-atom system interacting with the vacuum field can be written in the interaction picture as

$$\begin{aligned} H_I = \sum_{\alpha=A,B} \Big(& \sum_{ks} d_{eg}^\alpha \sigma_{eg}^\alpha a_{ks} e^{i[\vec{k}\cdot\vec{x}_\alpha + (\omega_1 - \omega_{ks})t]} \\ & + \sum_{k's'} d_{\mu g}^\alpha \sigma_{\mu g}^\alpha a_{k's'} e^{i[\vec{k'}\cdot\vec{x}_\alpha + (\omega_2 - \omega_{k's'})t]} + H.c. \Big), \end{aligned} \tag{1}$$

where, the vacuum Rabi coupling coefficients corresponding to atom α are

$$\begin{aligned} d_{eg}^\alpha &= i \left(\frac{2\pi\hbar\omega_{ks}}{V} \right)^{1/2} \vec{\wp}_{eg}^\alpha \cdot \vec{\varepsilon}_{ks} \\ \text{and} \quad d_{\mu g}^\alpha &= i \left(\frac{2\pi\hbar\omega_{k's'}}{V} \right)^{1/2} \vec{\wp}_{\mu g}^\alpha \cdot \vec{\varepsilon}_{k's'}, \end{aligned} \tag{2}$$

and $\vec{\wp}_{eg}^\alpha$ ($\vec{\wp}_{\mu g}^\alpha$) is the dipole matrix element corresponding to the transition operator $\sigma_{eg}^\alpha = |e_\alpha\rangle\langle g_\alpha|$ ($\sigma_{\mu g}^\alpha = |\mu_\alpha\rangle\langle g_\alpha|$), $\vec{\varepsilon}_{ks}(\vec{\varepsilon}_{k's'})$ is the unit polarization vector of the vacuum mode with frequency ω_{ks} ($\omega_{k's'}$), a_{ks} ($a_{k's'}$) is the photon annihilation operator corresponding to the vacuum field with wave vector k (k') and polarization s (s'), ω_1 (ω_2) is the atomic frequency corresponding to $|e_\alpha\rangle \leftrightarrow |g_\alpha\rangle$ ($|\mu_\alpha\rangle \leftrightarrow |g_\alpha\rangle$) transitions, and x_α is the position of the atom α.

We use the Zwanzig projection operator method [22, 23] to trace over the field degrees of freedom and obtain a reduced density matrix equation for the atoms. We use the Born and Markoff approximation to obtain a memoryless master equation. Referring to Ref. [19] and without duplicating the lengthy calculation, we write the reduced density matrix equation for the atoms as

$$\dot{\rho} = -i[\mathcal{V}_{dd}, \rho] + (\mathcal{L}_s + \mathcal{L}_{dd})\rho, \tag{3}$$

where

$$\mathcal{V}_{dd} = G_1 \sigma^A_{eg} \otimes \sigma^B_{ge} + G_2 \sigma^A_{\mu g} \otimes \sigma^B_{g\mu} + H.c., \tag{4}$$

is the part of the dd-interaction that contributes to the level shift. The coupling coefficients are

$$\begin{aligned} G_1 &= \sum_{ks} \frac{\pi}{\hbar^2(\omega_1 - \omega_{ks})} d^A_{eg} d^B_{ge} e^{i\vec{k}\cdot\vec{R}}, \\ G_2 &= \sum_{ql} \frac{\pi}{\hbar^2(\omega_2 - \omega_{ql})} d^A_{\mu g} d^B_{g\mu} e^{i\vec{k}\cdot\vec{R}}. \end{aligned} \tag{5}$$

Here $\vec{R} = \vec{x}_A - \vec{x}_B$. Further, the Liouvillian operators $\mathcal{L}_j$ are:

$$\begin{aligned} \mathcal{L}_s\rho = {} & \gamma_1[(2\sigma^A_{ge}\rho\sigma^A_{eg} - \sigma^A_{ee}\rho - \rho\sigma^A_{ee}) + A \to B] \\ & + \gamma_2[(2\sigma^A_{g\mu}\rho\sigma^A_{\mu g} - \sigma^A_{\mu\mu}\rho - \rho\sigma^A_{\mu\mu}) + A \to B], \end{aligned} \tag{6}$$

corresponding to spontaneous emission of the atoms, and

$$\begin{aligned} \mathcal{L}_{dd}\rho = {} & [\Gamma_1(2\sigma^B_{ge}\rho\sigma^A_{eg} - \sigma^A_{eg}\sigma^B_{ge}\rho - \rho\sigma^A_{eg}\sigma^B_{ge}) + H.c.] \\ & + [\Gamma_2(2\sigma^B_{g\mu}\rho\sigma^A_{\mu g} - \sigma^A_{\mu g}\sigma^B_{g\mu}\rho - \rho\sigma^A_{\mu g}\sigma^B_{g\mu}) + H.c.], \end{aligned} \tag{7}$$

corresponding to the dd coupling mediated by the vacuum. Note that the subscript in ρ_a is dropped for brevity. Here the spontaneous decay rates are given as

$$\begin{aligned} \gamma_1 &= \frac{1}{\hbar^2} \sum_{ks} \pi\delta(\omega_1 - \omega_{ks}) |d_{eg}|^2, \\ \gamma_2 &= \frac{1}{\hbar^2} \sum_{ql} \pi\delta(\omega_2 - \omega_{ql}) |d_{\mu g}|^2, \end{aligned} \tag{8}$$

and the atom-atom coupling coefficients are obtained as

$$\begin{aligned} \Gamma_1 &= \frac{1}{\hbar^2} \sum_{ks} \pi\delta(\omega_1 - \omega_{ks}) |d_{eg}|^2 e^{i\vec{k}\cdot\vec{R}}, \\ \Gamma_2 &= \frac{1}{\hbar^2} \sum_{ks} \pi\delta(\omega_2 - \omega_{ql}) |d_{\mu g}|^2 e^{i\vec{k}\cdot\vec{R}}. \end{aligned} \tag{9}$$

Further the index α has been dropped as we consider that the atomic dipoles corresponding to the same atomic transitions are parallel to each other, i.e., $\vec{\wp}_{ij}^{A} \parallel \vec{\wp}_{ij}^{B}$. Clearly, the radiative coupling terms Γ_i and G_i have numerical significance only in the limit $|kR|$ and $|k'R| \leq 1$. In the other limit, when the interatomic distance R is too large, only the spontaneous emission terms survive and the atoms behave as two independent atoms. We refer to [19] for the detail steps of the calculation.

Now let us assume that initially, the two-atom state is $|e\mu\rangle$, where the notation $|ij\rangle \equiv |i_A\rangle \otimes |j_B\rangle$, $i, j = e, \mu, g$. The nine two-atom basis states are $|ee\rangle$, $|e\mu\rangle$, $|eg\rangle$, $|\mu e\rangle$, $|\mu\mu\rangle$, $|\mu g\rangle$, $|ge\rangle$, $|g\mu\rangle$ and $|gg\rangle$, and we number them 1 through 9 in the same order as above to simplify the notations for the density matrix elements $\langle i_A j_B|\rho|i'_A j'_B\rangle$. For examples, the density matrix element $\langle e_A \mu_B|\rho|e_A \mu_B\rangle$ corresponding to our initial state $|e\mu\rangle$ is represented as ρ_{22} in the new notation. The full density matrix equation involves 81 matrix elements but for the above initial condition, many elements would be identically zero and only 10 density matrix elements survive. We consider the geometry where dipole matrix elements are orthogonal to each other and are real (as discussed in Sec. III of [19]), such that the parameters G_i, γ_i, Γ_i are real numbers seen Eq. (26) in [19]. In the following, we explicitely write the dynamics equations only for those surviving density matrix elements as

$$\begin{aligned}
\dot{\rho}_{22} &= -2(\gamma_1 + \gamma_2)\rho_{22}, \\
\dot{\rho}_{33} &= -2\gamma_1\rho_{33} + 2\gamma_2\rho_{22} - \Gamma_1(\rho_{73} + \rho_{37}) \\
&\quad -iG_1(\rho_{73} - \rho_{37}), \\
\dot{\rho}_{37} &= -2\gamma_1\rho_{37} - \Gamma_1(\rho_{77} + \rho_{33}) - iG_1(\rho_{77} - \rho_{33}), \\
\dot{\rho}_{66} &= -2\gamma_2\rho_{66} - \Gamma_2(\rho_{86} + \rho_{68}) - iG_2(\rho_{86} - \rho_{68}), \\
\dot{\rho}_{68} &= -2\gamma_2\rho_{68} - \Gamma_2(\rho_{88} + \rho_{66}) - iG_2(\rho_{88} - \rho_{66}), \\
\dot{\rho}_{77} &= -2\gamma_1\rho_{77} - \Gamma_1(\rho_{37} + \rho_{73}) - iG_1(\rho_{37} - \rho_{73}), \\
\dot{\rho}_{88} &= -2\gamma_2\rho_{88} + 2\gamma_1\rho_{22} - \Gamma_2(\rho_{86} + \rho_{68}) \\
&\quad -iG_2(\rho_{68} - \rho_{86}), \\
\dot{\rho}_{99} &= 2\gamma_1(\rho_{33} + \rho_{77}) + 2\gamma_2(\rho_{66} + \rho_{88}) \\
&\quad +2\Gamma_1(\rho_{37} + \rho_{73}) + 2\Gamma_2(\rho_{68} + \rho_{86}).
\end{aligned} \tag{10}$$

Note that the conjugate matrix elements ρ_{73} and ρ_{86} (conjugates of ρ_{37} and ρ_{68}, respectively) also evolve. Using the Laplace transform method, we solve the above coupled equations for the density matrix elements with the initial condition $\rho_{22} = 1$ to obtain their time evolution as

$$
\begin{aligned}
\rho_{22}(t) &= e^{-2(\gamma_1+\gamma_2)t}, \\
\rho_{33}(t) &= \frac{\gamma_2 e^{-2\gamma_1 t}}{2} \times \\
&\Big[\frac{\gamma_2\cosh(2\Gamma_1 t) - \Gamma_1 \sinh(2\Gamma_1 t) - \gamma_2 e^{-2\gamma_2 t}}{\gamma_2^2 - \Gamma_1^2} \\
&+ \frac{\gamma_2 \cos(2G_1 t) + G_1 \sin(2G_1 t) - \gamma_2 e^{-2\gamma_2 t}}{\gamma_2^2 + G_1^2}\Big], \\
\rho_{37}(t) &= \frac{\gamma_2 e^{-2\gamma_1 t}}{2} \times \\
&\Big[\frac{\Gamma_1 e^{-2\gamma_2 t} - \Gamma_1\cosh(2\Gamma_1 t) + \gamma_2 \sinh(2\Gamma_1 t)}{\Gamma_1^2 - \gamma_2^2} \\
&- \frac{iG_1 \cos(2G_1 t) - i\gamma_2 \sin(2G_1 t) - iG_1 e^{-2\gamma_2 t}}{G_1^2 + \gamma_2^2}\Big], \\
\rho_{66}(t) &= \frac{\gamma_1 e^{-2\gamma_2 t}}{2} \times \\
&\Big[\frac{\gamma_1\cosh(2\Gamma_2 t) - \Gamma_2 \sinh(2\Gamma_2 t) - \gamma_1 e^{-2\gamma_1 t}}{\gamma_1^2 - \Gamma_2^2} \\
&- \frac{\gamma_1 \cos(2G_2 t) + G_2 \sin(2G_2 t) - \gamma_1 e^{-2\gamma_1 t}}{\gamma_1^2 + G_2^2}\Big] \\
\rho_{68}(t) &= \rho_{37}^*(t)\big|_{1\leftrightarrow 2}, \rho_{77}(t) = \rho_{66}(t)\big|_{1\leftrightarrow 2}, \\
\rho_{88}(t) &= \rho_{33}(t)\big|_{1\leftrightarrow 2}, \\
\rho_{99}(t) &= 1 - \rho_{33}(t) - \rho_{66}(t) - \rho_{77}(t) - \rho_{88}(t).
\end{aligned}
\tag{11}
$$

It may be noted that the initial state $\rho_{22} \equiv \langle e_A\mu_B|\rho|e_A\mu_B\rangle$ decays with a rate of the sum of the decays of both excited states but does not depend on the dd coupling terms Γ_i and G_i. However, the other population and cross terms strongly depend on the dd coupling. The dd terms Γ_i play the role of decays via the cosine and sine hyperbolic functions and the coefficients G_i cause the dd coupling induced vacuum Rabi oscillations. The time dependent solutions of the matrix elements show the oscillations with frequencies determined by the atom-atom coupling coefficients G_1 and G_2. Further, the dd terms are strongly dependent on the interatomic distance R. Hence the dynamics of the density matrix elements are also strongly affected by R. We will present numerical plots and discussions for some of the important density matrix elements that help in evolving the SGE in the following section.

III. TIME EVOLUTION OF SGE

In this section, we calculate the time evolution of the entanglement between two-atoms. Out of various different methods to calculate entanglement between the two atoms, we choose the negativity, defined by [24]

$$N(\rho) = \frac{\|\rho^{T_A}\| - 1}{2} = -\sum_i {}' \lambda_i, \tag{12}$$

as the measure of entanglement. Here $\|\rho^{T_A}\|$ denotes the trace norm of ρ^{T_A} [25]; ρ^{T_A} is the partial transposition matrix of the atomic system density operator $\rho(t)$. The primed sum in the above equation represents the sum over only the negative eigenvalues λ_i of ρ^{T_A}. For a high dimensional system, while a non-zero $N(\rho)$ is a sufficient condition to prove that a system entangled, but a null $N(\rho)$ does not necessarily qualify a system to become disentangled. From the definition, the negativity N can also be greater than 1. For different dimensions of the density matrix, the maximum value of N is different. For a two-atom three-level system, such as ours, the state $\Psi = \frac{1}{\sqrt{3}}[|ee\rangle + |\mu\mu\rangle + |gg\rangle$ is maximally entangled one with the negativity $N(\Psi) = 1$.

Thus to obtain negativity in our two-atom system, we first calculate the eigenvalues of ρ^{T_A} by using the density matrix $\rho(t)$. After a lengthy calculation, we obtain the exact eigenvalues as

$$\begin{aligned}
\lambda_1 &= \lambda_2 = 0, \lambda_3 = \rho_{22}(t), \lambda_4 = \rho_{33}(t), \\
\lambda_5 &= \rho_{66}(t), \lambda_6 = \rho_{77}(t), \lambda_7 = \rho_{88}(t), \\
\lambda_8 &= \frac{\rho_{99}(t)}{2} + \frac{1}{2}\sqrt{\rho_{99}^2(t) + 4[|\rho_{37}(t)|^2 + |\rho_{68}(t)|^2]}, \\
\lambda_9 &= \frac{\rho_{99}(t)}{2} - \frac{1}{2}\sqrt{\rho_{99}^2(t) + 4[|\rho_{37}(t)|^2 + |\rho_{68}(t)|^2]}.
\end{aligned} \tag{13}$$

It is clear that among all of the above eigenvalues, only λ_9 can become negative. Therefore, if we obtain a negative λ_9, it is sufficient to prove the occurence of the SGE. Clearly, λ_9 can be negetive only if at least one of the matrix element $\rho_{37} \equiv \langle eg|\rho|ge\rangle$ or $\rho_{68} \equiv \langle \mu g|\rho|g\mu\rangle$ is non-zero, i.e., if there is an exchange of at least one photon between two atoms. Thus it is clearly established that radiative coupling leads to SGE in the two-atom system. Because both ρ_{37} and ρ_{68} contribute to the generation of entanglement. Thus the three-level atoms offer us more channels to establish entanglement than two-level one.

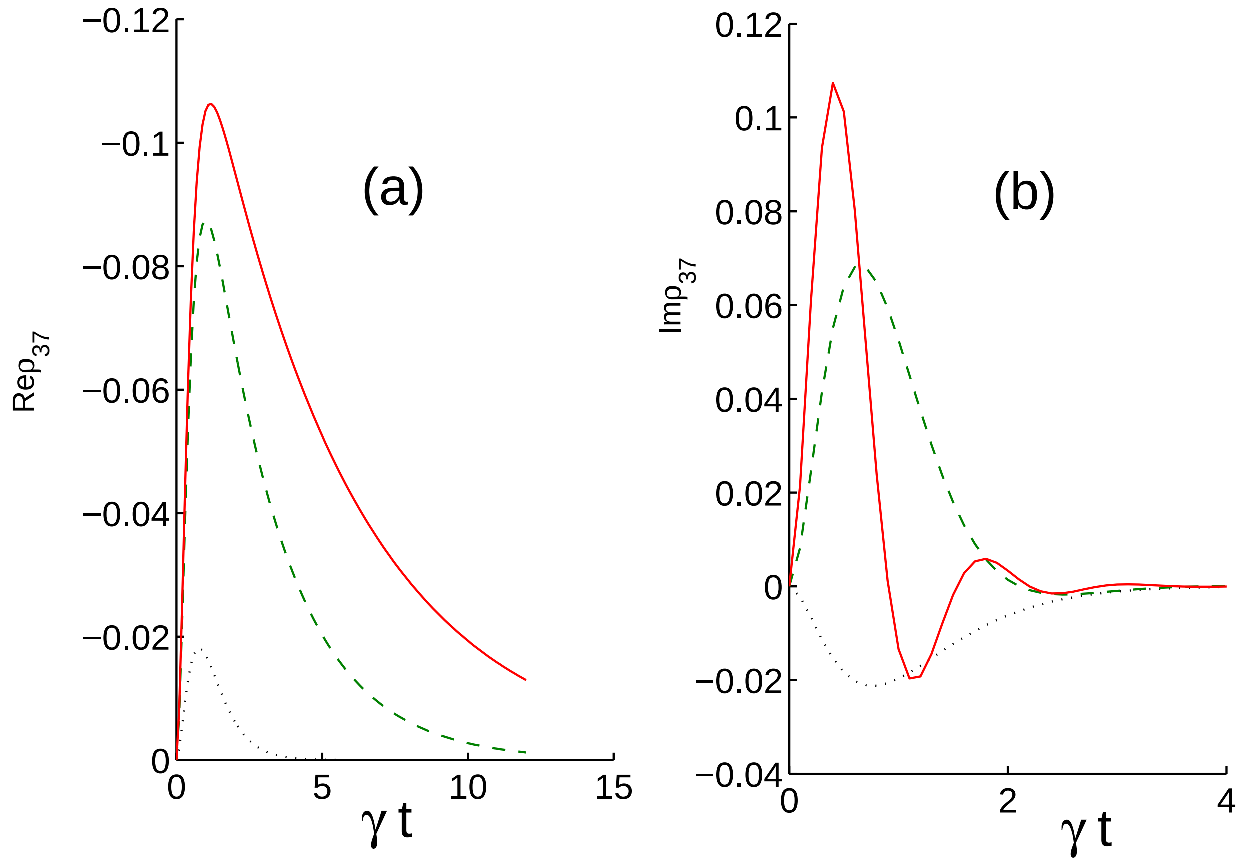

FIG. 2: (Color online) The time-dependence of real and imaginary parts of matrix element $\rho_{37}(t)$ with different values of $\Gamma_1(\Gamma_2)$ and $G_1(G_2)$. We set $\gamma = 1, r = 1.2$. The parameters $(G, \Gamma) \equiv (2.4, 0.9)$, $(0.9, 0.8)$, and $(-0.24, 0.2)$ correspond to $R = 0.83\lambda_1$, $1.18\lambda_1$ and $2.78\lambda_1$ that are represented in the figure as the solid, dashed and dotted line, respectively.

Before we proceed further, we first study the density matrix elements $\rho_{37}(t)$ and $\rho_{99}(t)$ that determine the entanglement. Note that the two excited states are non-degenerate. Assuming that $\omega_2 = r\omega_1$, and say $\gamma_1 = \gamma$; $\Gamma_1 = \Gamma$, $G_1 = G$, we have $\gamma_2 = r\gamma$, $\Gamma_2 = r\Gamma$ and $G_2 = rG$. For the numerical plots presented below, we use the parameters given in the Table I determined from their definitions in Eqs. (5, 8, 9) [30]. Note that as the interatomic distance reduces, Γ oscillates [19]. However, peak value of Γ decreases with larger interatomic distances due to decreased dipole-dipole coupling between the two atoms. Thus, Γ_i (G_i) can be same in two or more interatomic different distances. But any particular interatomic distance determines a particular value for the pair of $(\Gamma_i,\ G_i)$. We will concentrate on the the photon exchange process with decreased trend of Γ_i (G_i) with the increasing interatomic separation. So, the parameters given in table is approximately monotonic.

TABLE I: The interatomic distance and the corresponding coupling parameters. All the frequency units are scaled with γ.

R (in unit of λ_0)	Γ_1	Γ_2	G_1	G_2
0.50	$0.96\gamma_1$	$0.96\gamma_2$	$8.0\gamma_1$	$8.0\gamma_2$
0.83	$0.9\gamma_1$	$0.9\gamma_2$	$2.4\gamma_1$	$2.4\gamma_2$
1.18	$0.8\gamma_1$	$0.8\gamma_2$	$0.9\gamma_1$	$0.9\gamma_2$
2.78	$0.2\gamma_1$	$0.2\gamma_2$	$-0.24\gamma_1$	$-0.24\gamma_2$

In Fig. 2, we present the evolution of the density matrix element $\rho_{37} = \langle eg|\rho|ge\rangle$ representing the single photon radiative coupling process, in which atom A loses its excitation to excite atom B from its ground state $|g_A\rangle$ to the state $|e_B\rangle$. It is observed that, initially the process of exchange of photon increases. However, after reaching a maximum, Reρ_{37} falls off quickly within one spontaneous emission cycle. The time needed to reach the maximum value for Reρ_{37} is determined by Γ^{-1}. The maximum of Imρ_{37} occurs at $t \sim \gamma^{-1}$. In long time limit both real and imaginary part of ρ_{37} vanish. Similar conclusions can be derived for the matrix element ρ_{68} which physically represents the simultaneous probability of two processes $|\mu_A\rangle \to |g_A\rangle$ and $|g_B\rangle \to |\mu_B\rangle$.

In Fig. 3 we present the evolution of the population in the state $|gg\rangle$, i.e., the matrix element $\rho_{99}(t)$. This plot also supports the physical process we described above. We can see that again large values of Γ slows down $\rho_{99}(t)$ to reach at its asymptotic value. In other words, the radiative coupling process survives longer. The steady state value is $\rho_{99} = 1$, i.e., both the atoms reach their ground states in the long time limit.

Next, we discuss the property of the entanglement generated this atomic system. We present the plot of $N(t)$ that describes the time evolution of the SGE for different values of Γ and G in Fig. 4. It is shown that at $t = 0$, there is no entanglement because initially the atomic system is in the state $|e\mu\rangle$. For $t > 0$ the entanglement evolves to reach its maximum value, and then undergoes a process of disentanglement. Finally, steady state of negativity becomes identically zero. But the relaxation time becomes longer when the atoms are nearer (seen solid line in Fig. 2a and Fig. 4). From the solution Eq.(11), we know that relaxation time of Re (ρ_{37}) is determined by $max\{\frac{1}{2(\gamma_1+\Gamma_1)}, \frac{1}{2(\gamma_1-\Gamma_1)}\}$ which

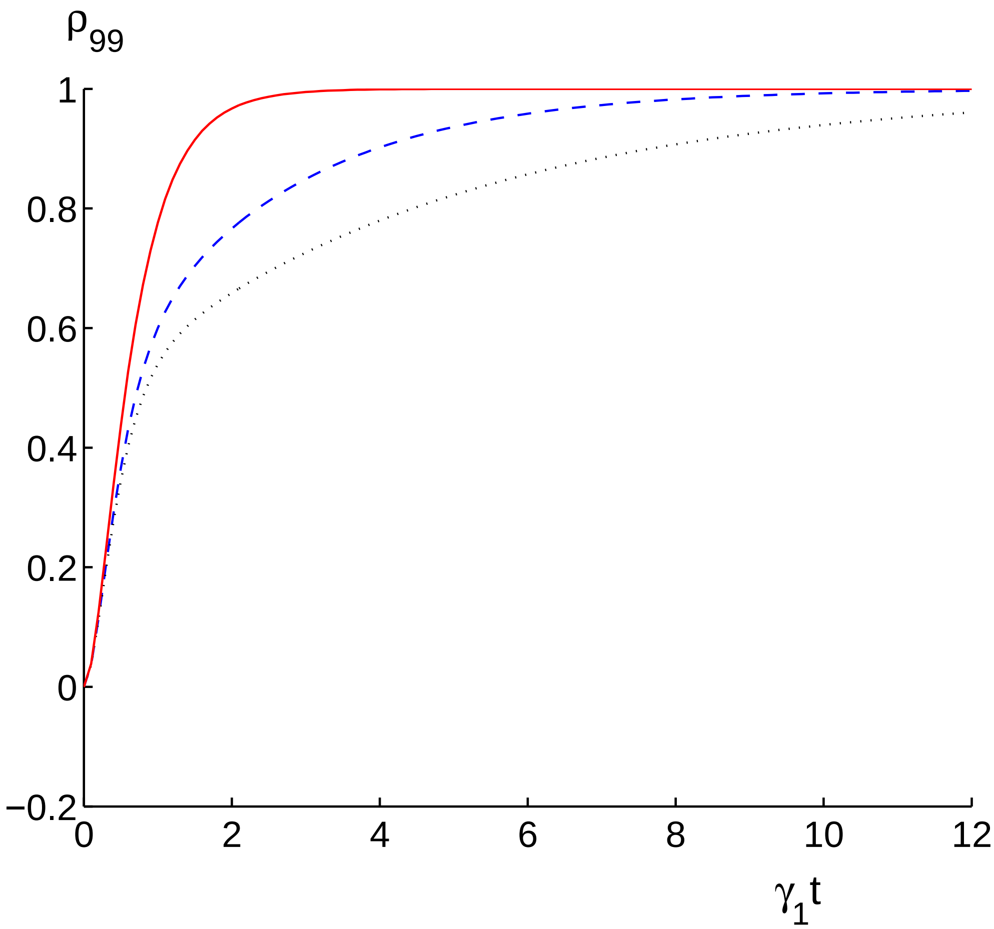

FIG. 3: (Color online) The matrix element for our atomic system as a function of t corresponding to different pairs of (G, Γ). The curves and also corresponding legends are same as in Fig.2.

means larger the value of $|\Gamma_1|$, longer the relaxation time for disentanglement. This is the competition between dd coupling and the all-direction spontaneous emission. The evolution of SGE presented here can be understood as due to the following multiple radiative exchange processes: spontaneous emission of atom A via $|e_A\rangle \rightarrow |g_A\rangle$ (atom B via $|\mu_B\rangle \rightarrow |g_B\rangle$) transition is followed by an absorption of the spontaneously emitted radiation by atom B via $|g_B\rangle \rightarrow |e_B\rangle$ (atom A via $|g_A\rangle \rightarrow |\mu_A\rangle$), as it is expressed by $\rho_{37}(t)$ $(\rho_{68}(t))$ causing the two-atom entanglement.

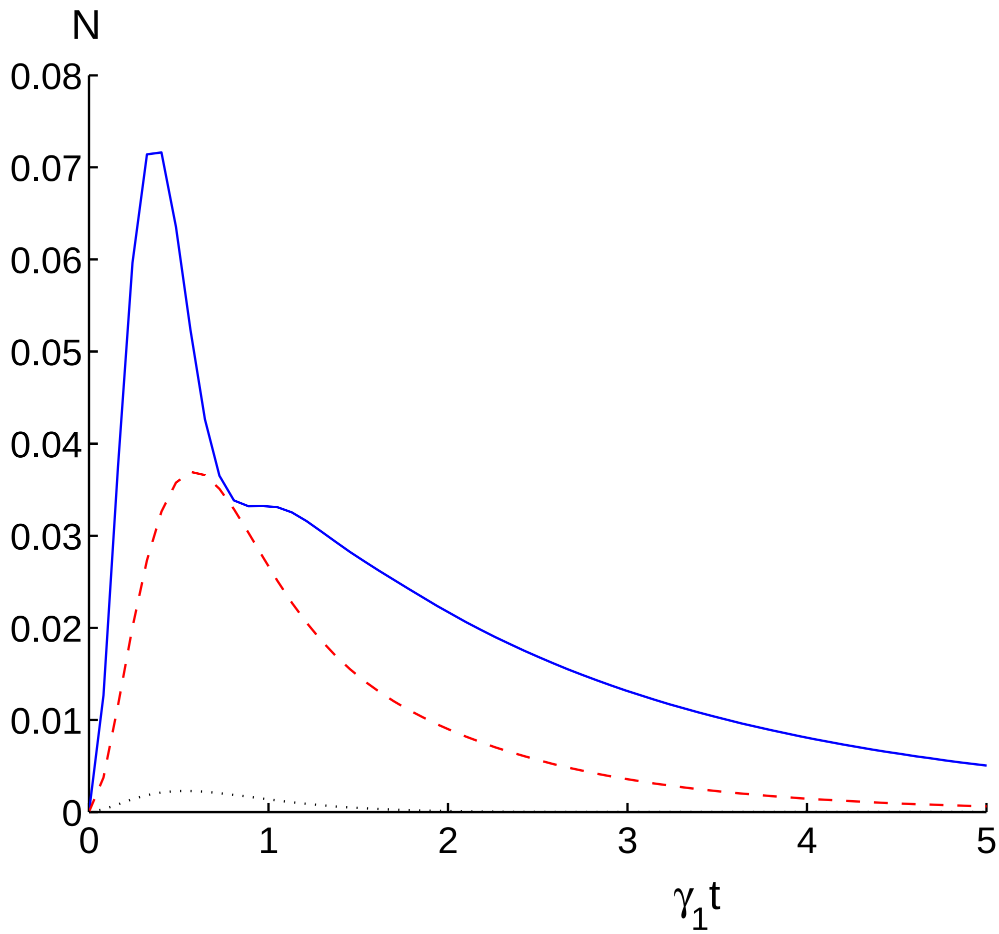

FIG. 4: (Color online) The negativity for our atomic system as a function of t corresponding to different pairs of (G, Γ). All the parameters correspond to Fig.2, respectively.

IV. STEADY STATE ENTANGLEMENT WITH BROADBAND INCOHERENT PUMPING

From the previous section, we have seen that the disadvantage of SGE is the temporal evolution and the quickly diminishing of SGE due to the decays in the system. However, for any useful application, for example, to use the two-atom system as a coupled qubit, sustaining the generated SGE is essential. In order to achieve a steady state entanglement, we introduce an incoherent pump to continually repump population to the excited states. Steady state entanglement using a classical coherent pumping is discussed in [26] where the coherent pumping with single frequency drive the atoms. Coherent field assisted entanglement is rather intutive. On the contrary, the usual notion associated with an incoherent pumping with a decoherence process and hence a source of disentanglement. However, in what follows below, we show that an appropriate strength of incoherent pump can lead to a steady entanglement between the two atoms.

We consider a broadband incoherent pump acting on the two atoms which incoherently drive the population from $|g\rangle$ to $|e\rangle$ ($|\mu\rangle$) levels. Incoherent pumping can be modeled as an inverse process of spontaneous emission [27, 28]. Thus we add a third Liouvillian $\mathcal{L}_{\rm inc}$ to the the master equation Eq.(4) to get

$$\dot{\rho} = -i[\mathcal{V}_{dd}, \rho] + (\mathcal{L}_s + \mathcal{L}_{dd} + \mathcal{L}_{\rm inc})\rho, \tag{14}$$

where

$$\begin{aligned} \mathcal{L}_{\rm inc}\rho \;=\; & \Lambda_1[(2\sigma^A_{eg}\rho\sigma^A_{ge} - \sigma^A_{gg}\rho - \rho\sigma^A_{gg}) + A \to B] \\ & +\Lambda_2[(2\sigma^A_{\mu g}\rho\sigma^A_{g\mu} - \sigma^A_{\mu\mu}\rho - \rho\sigma^A_{\mu\mu}) + A \to B]. \end{aligned} \tag{15}$$

where Λ_1 and Λ_2 denote incoherent pumping rates for $|g_\alpha\rangle \to |e_\alpha\rangle$ and $|g_\alpha\rangle \to |\mu_\alpha\rangle$ transitions respectively. We explicitly write the density matrix equations involved in presence of the incoherent pump as

$$
\begin{aligned}
\dot{\rho}_{11} &= -4\gamma_1\rho_{11} + 2\Lambda_1(\rho_{77}+\rho_{33}) \\
\dot{\rho}_{22} &= -2(\gamma_1+\gamma_2)\rho_{22} + 2\Lambda_1\rho_{88} + 2\Lambda_2\rho_{33}, \\
\dot{\rho}_{33} &= -2s_1\rho_{33} + 2\gamma_2\rho_{22} + 2\gamma_1\rho_{11} + 2\Lambda_1\rho_{99} \\
&\quad -iG_1(\rho_{73}-\rho_{37}) - \Gamma_1(\rho_{73}+\rho_{37}), \\
\dot{\rho}_{37} &= -2s_1\rho_{37} - \Gamma_1(\rho_{77}+\rho_{33}) \\
&\quad +2\Gamma_1\rho_{11} - iG_1(\rho_{77}-\rho_{33}), \\
\dot{\rho}_{44} &= -2(\gamma_1+\gamma_2)\rho_{44} + 2\Lambda_1\rho_{66} + 2\Lambda_2\rho_{77} \\
\dot{\rho}_{55} &= -4\gamma_2\rho_{55} + 2\Lambda_2(\rho_{66}+\rho_{88}) \\
\dot{\rho}_{66} &= -2s_2\rho_{66} + 2\gamma_1\rho_{44} + 2\gamma_2\rho_{55} + 2\Lambda_2\rho_{99} \\
&\quad -iG_2(\rho_{86}-\rho_{68}) - \Gamma_2(\rho_{86}+\rho_{68}), \\
\dot{\rho}_{68} &= -2s_2\rho_{68} - \Gamma_2(\rho_{88}+\rho_{66}) + 2\Gamma_1\rho_{55} - iG_2(\rho_{88}-\rho_{66}), \\
\dot{\rho}_{77} &= -2s_1\rho_{77} + 2\gamma_1\rho_{11} + 2\gamma_2\rho_{44} + 2\Lambda_1\rho_{99} \\
&\quad -\Gamma_1(\rho_{37}+\rho_{73}) - iG_1(\rho_{37}-\rho_{73}), \\
\dot{\rho}_{88} &= -2s_2\rho_{88} + 2\gamma_1\rho_{22} + 2\gamma_2\rho_{55} + 2\Lambda_2\rho_{99} \\
&\quad -\Gamma_2(\rho_{86}+\rho_{68}) - iG_2(\rho_{68}-\rho_{86}), \\
\dot{\rho}_{99} &= 2\gamma_1(\rho_{33}+\rho_{77}) + 2\gamma_2(\rho_{66}+\rho_{88}) - 4(\Lambda_1+\Lambda_2)\rho_{99} \\
&\quad +2\Gamma_1(\rho_{37}+\rho_{73}) + 2\Gamma_2(\rho_{68}+\rho_{86}).
\end{aligned}
\tag{16}
$$

where $s_\alpha = \gamma_\alpha + \Lambda_1 + \Lambda_2$ ($\alpha = 1, 2$). Note that in presence of the incoherent pumping Λ_i, the non-zero additional terms are the populations $|ee\rangle$, $|\mu\mu\rangle$, $|\mu e\rangle$. Hence we have a total of 13 non-vanishing density matrix elements in presence of Λ_i. Since we are looking for a steady state SGE, we calculate the steady state values of the matrix elements by setting the differentials in the left-hand sides as zero and solving the coupled equations. The analytical solutions that we obtain are

$$
\begin{aligned}
\rho_{11} &= \frac{\gamma_2}{b}\Lambda_1 a_2, \\
\rho_{22} &= \frac{\gamma_1\gamma_2}{b(\gamma_1+\gamma_2)}(a_1\Lambda_1 + a_2\Lambda_2), \\
\rho_{33} &= \frac{\gamma_1\gamma_2}{b}a_2, \\
\rho_{37} &= \frac{\Gamma_1\gamma_2}{s_1 b}(\Lambda_1-\gamma_1)a_2, \quad \rho_{44} = \rho_{22}, \\
\rho_{55} &= \frac{\gamma_1 a_1}{b}\Lambda_2, \quad \rho_{66} = \frac{\gamma_1\gamma_2 a_1}{b}, \\
\rho_{68} &= \frac{\gamma_1\Gamma_2}{s_2 b}(\Lambda_2-\gamma_2)a_1, \\
\rho_{77} &= \rho_{33}, \quad \rho_{88} = \rho_{66}, \\
\rho_{99} &= \frac{2\gamma_1\gamma_2}{b}(\beta_1 a_2 + a_1\beta_2),
\end{aligned} \tag{17}
$$

with

$$
\begin{aligned}
a_1 &= \Lambda_2[\gamma_1 + 2\beta_1(\gamma_1+\gamma_2)], \\
a_2 &= \Lambda_1[\gamma_2 + 2\beta_2(\gamma_1+\gamma_2)], \\
\beta_j &= \frac{s_\alpha\gamma_\alpha^2 + \Gamma_\alpha^2(\Lambda_\alpha-\gamma_\alpha)}{2s_\alpha\gamma_\alpha(\Lambda_1+\Lambda_2)}, \ j=1,2, \\
b &= 2\gamma_1\gamma_2(\gamma_1+\gamma_2)[(\beta_1+1)a_2 + (\beta_2+1)a_1] \\
&\quad +(\gamma_1+\gamma_2)(a_2\gamma_2\Lambda_1 + a_1\gamma_1\Lambda_2) \\
&\quad +\gamma_1\gamma_2(2a_2\Lambda_2 + 2a_1\Lambda_1)
\end{aligned} \tag{18}
$$

It is interesting to note that the steady state of the density matrix elements do not depend on G_1 and G_2. The level shift parameters G_1 and G_2 typically contribute to oscillation of population and coherence terms. Hence, in the long time limit, such fast oscillation terms vanish. Thus steady state solutions in Eq. (17) are independent of G_1 and G_2.

Once again, as in the previous section, we calculate the eigenvalues of ρ^{T_A} to measure the entanglement. We obtain equation for the non-zero eigen values as

$$
\begin{aligned}
&(\rho_{11}-\lambda)(\rho_{55}-\lambda)(\rho_{99}-\lambda) - |\rho_{37}|^2(\rho_{55}-\lambda) \\
&-|\rho_{68}|^2(\rho_{11}-\lambda) = 0.
\end{aligned} \tag{19}
$$

We obtain the numerical values of λ solving the above equation and substitute in Eq. (12) to obtain the steady state negativity as a function of Λ_i, as shown in Fig. 5. We have scaled the incoherent pumping rate Λ_i with the spontaneous decay rate $\gamma_1 = \gamma$ and also for simplicity we have assumed $\Lambda_1 = \Lambda_2 = \Lambda$. Clearly, a non-zero steady state entanglement is obtained by incoherently repumping the excited state. Smaller the interatomic distance, larger is the steady state entanglement. Further, as the incoherent pumping rate is increased, the SGE increases but after reaching a certain optimal value at around $\Lambda = 0.08\gamma$, the atomic entanglement starts to reduce. For smaller the interatomic distances, even stronger incoherent pumping can be used to get entangled atoms. Without any incoherent pumping the steady state SGE is identically zero. Physically, the increase in entanglement with the incoherent pumping can be understood as follows: the spontaneous emission in either of the two atoms followed by exchange of photon between them generates SGE. But that does not survive longer because the spontaneously emitted photon can escape in any arbitrary direction. Once both atoms loose their excitation, SGE vanishes. An incoherent pump assists the atoms to bring back to the desirable excitation so that more spontaneous emissions and hence photon exchanges can take place between the two atoms. Thus, increasing the repumping via incoherent pumping helps increasing the SGE. However, incoherent repumping also competes with the photon exchange process to re-excite the atoms. While an excitation due to the photon-exchange process enhances the entanglement, an excitation by incoherent process has no direct contribution to the entanglement. In fact, for a larger Λ, the incoherent excitation dominates the photon exchange process and hence causes a decrease in SGE. For $\Lambda \gg \Gamma_i,\ G_i$, SGE becomes identically zero.

V. DISCUSSION AND SUMMARY

We have investigated the spontaneously generated entanglement in a system of two three-level atoms are coupled to the common vacuum field. We have presented the time evolution of SGE due to the photon exchange between the two atoms. We have shown that both the magnitude of entanglement and the survival period of SGE are enhanced by reducing the interatomic distance. From our analytical calculations, we have shown the strong dependence of the SGE on the radiative coupling parameters. We have explicitly demonstrated that the multilevel atoms are preferable compared to their two-level counterparts for SGE, because each channel adds to enhance the magnitude of the entanglement. In the long time limit, however, SGE vanishes.

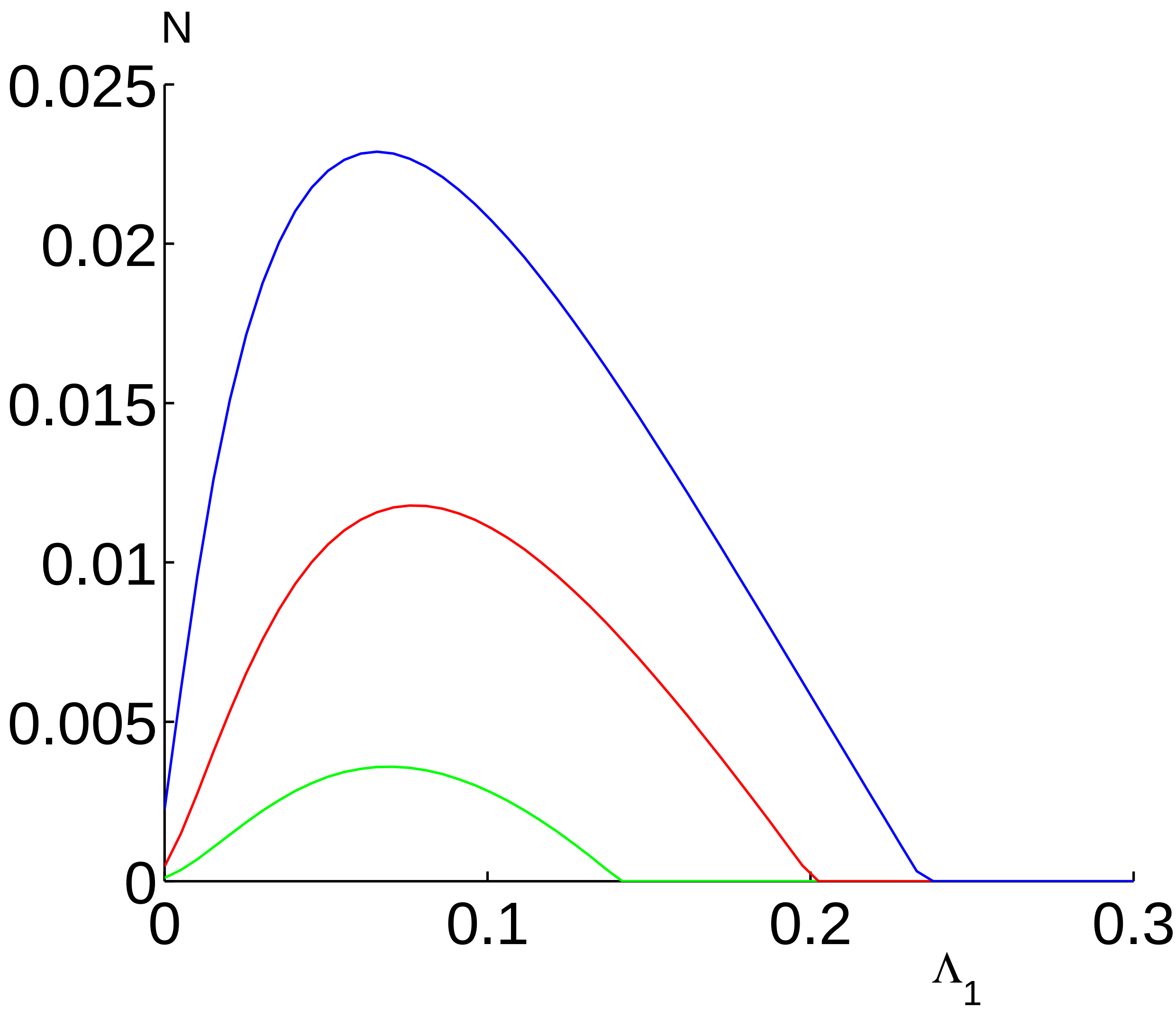

FIG. 5: (Color online) The steady state negativity for our atomic system as a function of Λ, where we set $\Lambda_1 = \Lambda_2 = \Lambda$, $\gamma_1 = \gamma = 1$, $r = 1.2$, $\gamma_2 = r\gamma$. From bottom to top, the values of $\Gamma = 0.8, 0.9$ and 0.96, corresponding to $R = 1.18\lambda_1$, $0.83\lambda_1$ and $0.5\lambda_1$, respectively.

Further, to reinforce the above short term evolution of SGE in the radiatively-coupled two-atom system, we have proposed to use an incoherent pump that assists in repumping the deexcited atoms and sustain the SGE. We have demonstrated that for a certain range of incoherent pumping, the steady state value of SGE increases as it prevents atoms from loosing their excited state population. However, since incoherent pumping competes with the two-atom photon exchange process to reexcite the atoms, a stronger incoherent pumping is shown to be undesirable. We have shown that an appropriate rate of incoherent pump can help producing optimal SGE.

The above entanglement can further increase (not discussed here) if one considers atoms having degenerate or near degenerate excited states in their excited state that has additional coherences [19], which will be discussed elsewhere. The radiative coupling discussed above can be realized in any a tight ion trap. However, this work can be generalized to realizing SGE in a chain of quantum dots or even in a typical dense multi-atom system. We believe this work will open up a new way to utilize the naturally occurring SGE to realize an efficient entanglement source.

This work was supported by NSFC under Grant No.10774020, and also supported by SRF for ROCS, SEM. AKP is indebted to Prof. G. S. Agarwal, Dr. P. Anisomov, and Prof. M. O. Scully for discussions on various aspects of the two-atom multilevel systems.

---

[1] B. B. Blinov, D. L. Moehring, L.-M. Duan, and C. Monroe, Nature **428**, 153 (2004).

[2] Y. Miroshnychenko, W. Alt, I. Dotsenko, L. Fster, M. Khudaverdyan, D. Meschede, D. Schrader, A. Rauschenbeutel, Nature **442**, 151 (2006).

[3] H. Hfner, W. Hnsel, C. F. Roos, J. Benhelm, D. Chek-al-kar, M. Chwalla, T. Krber, U. D. Rapol, M. Riebe, P. O. Schmidt, C. Becher, O. Ghne, W. Dr, and R. Blatt, Nature **438**, 643 (2005).

[4] S. Haroche and J.-M. Raimond, *Exploring the Quantum*, Oxford Univ. Press, NY (2006); M. B. Plenio and S. F. Huelga, Phys. Rev. Lett. **88**, 197901 (2002); M. karja, N. M. Bornik, M. Lffler, and H. Walther, Phys. Rev. A **60**, 3229 (1999).

[5] J. I. Cirac and P. Zoller, Phys. Rev. A **50**, R2799 (1994); E. Hagley, X. Maître, G. Nogues, C. Wunderlich, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. **79**, 1 (1997); M. B. Plenio, S. F. Huelga, A. Beige and P. L. Knight, Phys. Rev. A **59**, 2468 (1999); A. Beige, S.Bose, D.Braun, S. F. Huelga, P. L. Knight, M. B. Plenio and V. Vedral, J. of Mod. Opt. **47**, 2583 (2000).

[6] G. K. Brennen, I. H. Deutsch, and P. S. Jessen, Phys. rev. A **61**, 062309 (2000).

[7] O. Mandel, M. Greiner, A. Widera, T. Rom, T. W. Hsch, and I. Bloch, Nature **425**, 937 (2003).

[8] C. Cabrillo, J. I. Cirac, P. Garc-Fernn dez, and P. Zoller, Phys. Rev. A 59, 1025 - 1033 (1999); zgr kir, Alexander A. Klyachko, and Alexander S. Shumovsky, Phys. Rev. A 71, 034303

(2005).

[9] M. Kiffner, J. Evers, and C. H. Keitel, Phys. Rev. A 75, 032313 (2007).

[10] L.-M. Duan, M. D. Lukin, J. I. Cirac, and P. Zoller, Nature (London) **414**, 513 (2001); C. W. Chou, H. de Riedmatten, D. Felinto, S. V. Polyakov, S. J. van Enk, and H. J. Kimble1, Nature (London) **438**, 828 (2005).

[11] M. O. Scully, Laser Phys. **17**, 635 (2007).

[12] M.O. Scully, E. S. Fry, C. H. Raymond Ooi and K. Wódkiewicz, Phys. Rev. Lett. **96**, 010501 (2006).

[13] U. Akram, Z. Ficek, and S. Swain, Phys. Rev. A **62**, 013413 (2000); L. Jakobezyk, J. Phys. A: Math. Gen. **35**, 6383 (2002); Z. Ficek and R. Tanaś, Phys. Rev. A **74**, 024304 (2006).

[14] M. S. Kim, J. Lee, D. Ahn, and P. L. Knight, Phys. Rev. A **65**, 040101 (2002); S. Schneider and G. J. Milburn, Phys. Rev. A **65**, 042107 (2002); J. Shang and H. Yu, Phys. Rev. A **75**, 012101 (2007).

[15] D. Rossini, T. Calarco, V. Giovannetti, S. Montangero, and R. Fazio, Phys. Rev. A **75**, 032333 (2007).

[16] B. Duplantier, J.-M. Raimond, V. Rivasseau (Eds.), *Quantum Decoherence, Progress in Mathematical Physics*, Vol. 48, Springer Verlag (2007).

[17] L. Derkacz and L. Jakóbczyk, Phys. Rev. A **61**, 062309 (2000).

[18] M. Kiffner, J. Evers, and C. H. Keitel, Phys. Rev. A **75**, 032313 (2007).

[19] G. S. Agarwal and A. K. Patnaik, Phys. Rev. A **63**, 043805 (2001).

[20] J. Evers, M. Kiffner, M. Macovei, and C. H. Keitel, Phys. Rev. A **73**, 023804 (2006); M. Macovei, Z. Ficek, and C. H. Keitel, Phys. Rev. A **73**, 063821 (2006).

[21] M. Kiffner, J. Evers, and C. H. Keitel, Phys. Rev. A **76**, 013807 (2007).

[22] L. Mandel and E. Wolf, *Optical coherence and quantum optics*, Cambridge Univ. Press (1995).

[23] G. S. Agarwal, *Quantum Statistical Theories of Spontaneous Emission and Their Relation to Other Approaches, Spring Tracts in Modern Physics: Quantum Optics* (Springer-Verlag, Berlin, 1974)

[24] G. Vidal, R. F. Werner, Phys. Rev. A **65**, 032314 (2002).

[25] K. Zyczkowski, P. Horodecki, A. Sanpera and M. Lewenstein, Phys. Rev. A **58**, 883 (1998).

[26] Ö. Cakir,A. A. Klyachko, and S. Shumovsky, Phys. Rev. A 71,034303(2005).

[27] Y. Mu and C. M. Savage, Phys. Rev. A, 46, 5944 (1992).

[28] T. Salzburger and H. Ritsch, Phys. Rev. Lett. 93,063002(2004).

[29] D. Bruβ and C. Macchiavello, Phys. Rev. Lett. **88**, 127901(2002); D. Kaszlikowski, P. Gnaciński, M. Żukowski, W. Miklaszewski, and A. Zeilinger, *ibid.* **85**, 4418(2000).

[30] The numerical parameters are obtained for a chosen geometry with $\theta = \frac{\pi}{2}$, $\phi = \frac{\pi}{4}$ in Eq. (26) of [19].